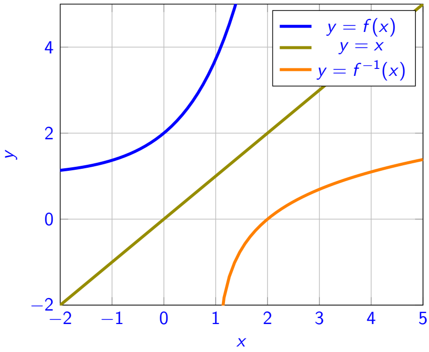
Chapter 3.8: Derivatives of Inverse Functions and Logarithms

Inverse Function



Observation: Tangent lines will also flip across the line $y = x$.

If $y = mx + b$ is a tangent line to $y = f(x)$ at $(a, f(a))$,

then $x = my + b$ is the tangent line to $y = f^{-1}(x)$ at $(f(a), a)$.

Note $x = my + b$ becomes $y = \frac{1}{m}(x - b)$.

The slope is a reciprocal: $\frac{1}{m}$.

Functions are related to their inverses by flipping across the line $y = x$.

Recall: f has to passes the horizontal line test to have inverse.

Derivatives of Inverse Functions

$$\boxed{\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}}$$

Derivation can be done using implicit function approach:

$$x = x$$
$$f(f^{-1}(x)) = x$$
$$\frac{d}{dx}\left[f(f^{-1}(x))\right] = \frac{d}{dx}\left[x\right]$$
$$f'(f^{-1}(x)) \cdot \frac{d}{dx}\left[f^{-1}(x)\right] = 1$$
$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}$$

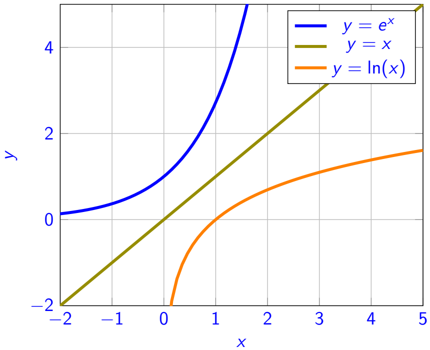
Example: $f(x) = x^3 + 4x + 5$.
Compute $\frac{d}{dx}\left(f^{-1}(x)\right)$ at $x = 10$.
We need to compute
$\frac{d}{dx}\left[f^{-1}(10)\right] = \frac{1}{f'(f^{-1}(10))}$.
We solve $f^{-1}(10)$ as

$$10 = x^3 + 4x + 5$$
$$0 = x^3 + 4x - 5$$
$$0 = (x-1)\cdot(x^2 + x + 5)$$

Hence $f^{-1}(10) = 1$.
Or you may guess $f^{-1}(10) = 1$:-)
Compute $f'(x) = 3x^2 + 4$.
Hence $f'(f^{-1}(10)) = f^{-1}(1) = 7$.

$$\frac{d}{dx}\left[f^{-1}(10)\right] = \frac{1}{f'(f^{-1}(10))} = \frac{1}{7}$$

Recall Exponential and Log Functions



$$f(x) = e^x \leftrightarrow f^{-1}(x) = \ln(x)$$

$$e^a e^b = e^{a+b} \leftrightarrow \ln(ab) = \ln(a) + \ln(b)$$

$$(e^a)^b = e^{ab} \leftrightarrow \ln\left(a^b\right) = b\ln(a)$$

$$\frac{e^a}{e^b} = e^{a-b} \leftrightarrow \ln\left(\frac{a}{b}\right) = \ln a - \ln b$$

$$e^{\ln x} = x \leftrightarrow \ln\left(e^x\right) = x$$

$$e^1 = e \leftrightarrow \ln(e) = 1$$

$$e^0 = 1 \leftrightarrow \ln(1) = 0$$

$$a = e^{\ln a} \qquad \log_a x = \frac{\ln x}{\ln a}$$

Derivative of $\ln(x)$

Using inverse derivative

$$\boxed{\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}}$$

$$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{e^{\ln(x)}}$$

$$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}$$

Using implicit function differentiation

$$\begin{aligned}
y &= \ln x \\
e^y &= x \\
\frac{d}{dx}\left(e^y\right) &= \frac{d}{dx}\left(x\right) \\
e^y \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{e^y} = \frac{1}{x}
\end{aligned}$$

$$\boxed{\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}}$$

Interesting: $\frac{d}{dx}\left[x^k\right] = kx^{k-1}$. How could you get x^{-1} on the right-hand side?

Examples for Derivatives with ln

1. $\dfrac{d}{dx}[\ln(3x)] = \dfrac{1}{3x} \cdot 3 = \dfrac{1}{x}$

2. $\dfrac{d}{dt}\left[t[\ln(t)]^2\right] = [\ln(t)]^2 + t \cdot 2[\ln(t)] \cdot \dfrac{1}{t}$

3. $\dfrac{d}{dx}\left[\dfrac{\ln(x)}{x}\right] = \dfrac{1/x \cdot x - \ln(x)}{x^2}$

4. $\dfrac{d}{dx}[3^x] = \dfrac{d}{dx}\left[e^{\ln(3^x)}\right] = \dfrac{d}{dx}\left[e^{x\ln(3)}\right] = e^{x\ln(3)}\ln(3) = \ln(3) \cdot 3^x$

5. $\dfrac{d}{dx}[x^x] =$
$$\frac{d}{dx}\left[\left(e^{\ln x}\right)^x\right] = \frac{d}{dx}\left[e^{x\ln x}\right] = e^{x\ln x}\left(1 \cdot \ln x + x \cdot \frac{1}{x}\right) = (1 + \ln x)\,x^x$$

Logarithmic Differentiation

Idea: Instead of using

$$y = f(x)$$
$$\frac{dy}{dx} = f'(x)$$

First take ln of both sides and then take the derivative.

$$\begin{aligned} y &= f(x) \\ \ln(y) &= \ln(f(x)) \\ \frac{1}{y}\frac{dy}{dx} &= \frac{d}{dx}\left[\ln(f(x)\right] \\ \frac{dy}{dx} &= y \cdot \frac{d}{dx}\left[\ln(f(x)]\right) \\ \frac{dy}{dx} &= f(x) \cdot \frac{d}{dx}\left[\ln(f(x)]\right) \end{aligned}$$

Example: $y = \sqrt{x(x+1)}$

$$\begin{aligned} y &= \sqrt{x(x+1)} \\ \ln(y) &= \ln(\sqrt{x(x+1)}) \\ \ln(y) &= \frac{1}{2}\ln(x(x+1)) \\ \ln(y) &= \frac{1}{2}[\ln(x) + \ln(x+1)] \\ \frac{d}{dx}\Big[\ln(y)\Big] &= \frac{d}{dx}\left[\frac{1}{2}[\ln(x) + \ln(x+1)]\right] \\ \frac{1}{y}\frac{dy}{dx} &= \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x+1}\right) \\ \frac{dy}{dx} &= \frac{y}{2}\left(\frac{1}{x} + \frac{1}{x+1}\right) \\ \frac{dy}{dx} &= \frac{\sqrt{x(x+1)}}{2}\left(\frac{1}{x} + \frac{1}{x+1}\right) \end{aligned}$$

Sometimes $\ln(f(x))$ is simpler for taking derivative than $f(x)$.

Logarithmic Differentiation Examples

Use logarithmic differentiation to compute the derivatives of following functions. Note we computed these already with different method.

$$y = 3^x$$

$$y = x^x$$

$$\begin{aligned}
y &= 3^x \\
\ln(y) &= \ln\left(3^x\right) \\
\ln(y) &= x\ln(3) \\
\frac{d}{dx}\Big[\ln(y)\Big] &= \frac{d}{dx}\left[x\ln(3)\right] \\
\frac{1}{y}\frac{dy}{dx} &= \ln(3) \\
\frac{dy}{dx} &= \ln(3)\cdot 3^x
\end{aligned}$$

$$\begin{aligned}
y &= x^x \\
\ln(y) &= \ln\left(x^x\right) \\
\ln(y) &= x\ln(x) \\
\frac{d}{dx}\Big[\ln(y)\Big] &= \frac{d}{dx}\left[x\ln(x)\right] \\
\frac{1}{y}\frac{dy}{dx} &= \ln(x) + x\cdot\frac{1}{x} \\
\frac{dy}{dx} &= y\cdot(\ln(x)+1) \\
\frac{dy}{dx} &= x^x\ln(x) + x^x
\end{aligned}$$

Logarithmic Differentiation Examples Round II

Use logarithmic differentiation to compute the derivatives of following:

$$y = \sqrt[3]{\frac{t}{t+1}} \qquad f(t) = t(t+1)(t+2)$$

$$\begin{aligned}
y &= \sqrt[3]{\frac{t}{t+1}} \\
\ln(y) &= \ln\left(\sqrt[3]{\frac{t}{t+1}}\right) \\
\ln(y) &= \frac{1}{3}[\ln(t) - \ln(t+1)] \\
\frac{d}{dx}\Big[\ln(y)\Big] &= \frac{d}{dx}\left[\frac{1}{3}[\ln(t) - \ln(t+1)]\right] \\
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{3}\left(\frac{1}{t} - \frac{1}{t+1}\right) \\
\frac{dy}{dx} &= \frac{1}{3} \cdot \sqrt[3]{\frac{t}{t+1}} \cdot \left(\frac{1}{t} - \frac{1}{t+1}\right)
\end{aligned}$$

$$\begin{aligned}
f(t) &= t(t+1)(t+2) \\
\ln(f(t)) &= \ln(t(t+1)(t+2)) \\
\ln(f(t)) &= \ln(t) + \ln(t+1) + \ln(t+2) \\
\frac{d}{dt}\Big[\ln(f(t))\Big] &= \frac{d}{dt}\Big[\ln(t) + \ln(t+1) + \ln(t+
\\
\frac{1}{f(t)}f'(t) &= \frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2} \\
f'(t) &= t(t+1)(t+2) \\
&\quad \cdot \left(\frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2}\right)
\end{aligned}$$

Chapter 3.8 Recap

$$\boxed{\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}}$$

$$\boxed{\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}}$$

If $y = f(x)$ then

$$\boxed{\frac{d}{dx}\left[\ln(y)\right] = \frac{d}{dx}\left[\ln(f(x)\right]}$$